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2)

$S_1 = 1 \quad S_2 = \frac{1}{2}$ $m_1 = 1, 0, -1$
$m_2 = \pm \frac{1}{2}.$

Note: The students do not need to do ... the CG coefficients

a) direct product basis: $|m_1 \; m_2\rangle$

$|1 \; \tfrac{1}{2}\rangle \quad |1 \; -\tfrac{1}{2}\rangle$
$|0 \; \tfrac{1}{2}\rangle \quad |0 \; -\tfrac{1}{2}\rangle$
$|-1 \; \tfrac{1}{2}\rangle \quad |-1 \; -\tfrac{1}{2}\rangle$

b)

FOR ANY OPERATOR, Q:

$$\frac{d}{dt}\langle Q\rangle_{(t)} = -\tfrac{i}{\hbar}\langle [Q, H]\rangle_{(t)}$$

For this case $H = H_0 + W$

$$\therefore \quad -\tfrac{i}{\hbar}\langle [S_{1x}, H]\rangle = -\tfrac{i}{\hbar}\langle [S_{1x}, H_0 + W]\rangle = -\tfrac{i}{\hbar}\langle [S_{1x}, W$$

$$W = a\vec{S}_1 \cdot \vec{S}_2 = a(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z})$$

$$\therefore \frac{d}{dt}\langle S_{1x}\rangle_t = \frac{-ia}{\hbar}\langle [S_{1x}, W]\rangle_t$$

$$[S_{1x}, W] = [S_{1x}, S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}]$$

$$= \underbrace{[S_{1x}, S_{1x}S_{2x}]}_{0} + [S_{1x}, S_{1y}S_{2y}] + [S_{1x}, S_{1z}S_{2z}]$$

$$= [S_{1x}, S_{1y}]S_{2y} + S_{1y}\underbrace{[S_{1x}, S_{2y}]}_{0} + [S_{1x}, S_{1z}]S_{2z} + S_{1z}[S_{1x},$$

$$= i\hbar S_{1z}S_{2y} + -i\hbar S_{1y}S_{2z}$$

$$= -i\hbar(\vec{S}_1 \times \vec{S}_2)_x$$

$$\therefore d/dt\langle S_{1x}\rangle_t = -a\langle(\vec{S}_1 \times \vec{S}_2)_x\rangle$$

c) Want the basis of eigenvectors for total angular moment.

6 possible $|J\ M\rangle$ (J: S_1+S_2, M: m_1+m_2)

$$\boxed{\begin{array}{ll} |\tfrac{3}{2}\ \tfrac{3}{2}\rangle & |\tfrac{1}{2}\ \tfrac{1}{2}\rangle \\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle & |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle & \\ |\tfrac{3}{2}\ -\tfrac{3}{2}\rangle & \end{array}}$$

These can be expressed in terms of the $|m_1\ m_2\rangle$ product basis from part a.

$$|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |1\ \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = |-1\ -\tfrac{1}{2}\rangle$$

For $|\tfrac32\ \tfrac12\rangle$ use lowering operator on $|\tfrac32\ \tfrac32\rangle$

$$J_-|\tfrac32\ \tfrac32\rangle = \hbar\sqrt{\tfrac32(\tfrac52)-\tfrac32(\tfrac12)}\,|\tfrac32\ \tfrac12\rangle = \sqrt3\,\hbar\,|\tfrac32\ \tfrac12\rangle$$

$$J_- = S_{1-}+S_{2-} \quad\Rightarrow\quad S_{1-}+S_{2-}|\tfrac32\ \tfrac32\rangle = \sqrt3\,\hbar\,|\tfrac32\ \tfrac12\rangle$$

$$\therefore\ |\tfrac32\ \tfrac12\rangle = \tfrac{1}{\sqrt3\hbar}(S_{1-}+S_{2-})|\tfrac32\ \tfrac32\rangle$$

$$= \tfrac{1}{\sqrt3\hbar}(S_{1-}+S_{2-})|1\ \tfrac12\rangle$$

$$= \tfrac{1}{\sqrt3\hbar}\Big[\hbar\sqrt{1(1+1)-1(1-1)}\,|0\ \tfrac12\rangle + \hbar\sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,|1$$

$$|\tfrac32\ \tfrac12\rangle = \sqrt{\tfrac23}\,|0\ \tfrac12\rangle + \tfrac1{\sqrt3}\,|1\ -\tfrac12\rangle$$

$$J_-|\tfrac32\ \tfrac12\rangle = \hbar\sqrt{\tfrac32(\tfrac52)-\tfrac12(-\tfrac12)}\,|\tfrac32\ -\tfrac12\rangle = 2\hbar\,|\tfrac32\ -\tfrac12\rangle$$

$$\therefore\ |\tfrac32\ -\tfrac12\rangle = \frac{1}{2\hbar}(S_{1-}+S_{2-})|\tfrac32\ \tfrac12\rangle$$

$$= \frac{1}{2\hbar}(S_{1-}+S_{2-})\Big[\sqrt{\tfrac23}\,|0\ \tfrac12\rangle + \tfrac1{\sqrt3}\,|1\ -\tfrac12\rangle\Big]$$

$$= \frac{1}{2\hbar}\Big\{\sqrt{\tfrac23}\Big(\hbar\sqrt{1(1+1)-0(0-1)}\,|-1\ \tfrac12\rangle + \hbar\sqrt{\tfrac12(1+\tfrac12)-\tfrac12(\tfrac12-1)}$$

$$+\frac{1}{\sqrt3}\Big(\hbar\sqrt{1(1+1)-1(1-1)}\,|0\ -\tfrac12\rangle + 0\Big)\Big\}$$

$$= \frac{1}{2\hbar}\Big\{\hbar\frac{2}{\sqrt3}\,|-1\ \tfrac12\rangle + \hbar\frac{\sqrt2}{\sqrt3}\,|0\ -\tfrac12\rangle + \hbar\frac{\sqrt2}{\sqrt3}\,|0\ -\tfrac12$$

$$= \frac{1}{\sqrt3}\,|-1\ \tfrac12\rangle + \frac{\sqrt2}{\sqrt3}\,|0\ -\tfrac12\rangle$$

For $J = \frac{1}{2}$.

$|\frac{1}{2}\ \frac{1}{2}\rangle$ must be a linear combination of the $|m_1\ m_2\rangle$ states which give $M = \frac{1}{2}$.

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = a\,|0\ \tfrac{1}{2}\rangle + b\,|1\ -\tfrac{1}{2}\rangle$$

From orthonormal condition : $a^2 + b^2 = 1$
& $\langle \frac{3}{2}\ \frac{1}{2} | \frac{1}{2}\ \frac{1}{2}\rangle = 0$

$$\Rightarrow \left(\tfrac{1}{\sqrt{3}}\langle 1\ -\tfrac{1}{2}| + \sqrt{\tfrac{2}{3}}\langle 0\ \tfrac{1}{2}|\right)\left(a|0\ \tfrac{1}{2}\rangle + b\,|1\ -\tfrac{1}{2}\rangle\right) = 0$$

$$\sqrt{\tfrac{2}{3}}\,a + \tfrac{1}{\sqrt{3}}\,b = 0$$

$$b = -\sqrt{2}\,a$$

so $\quad a^2 + b^2 = a^2 + 2a^2 = 1$

$$a = \frac{1}{\sqrt{3}} \Rightarrow b = -\sqrt{\frac{2}{3}}$$

so: $\quad |\frac{1}{2}\ \frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|0\ \frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|1\ -\frac{1}{2}\rangle$

To get $|\frac{1}{2}\ -\frac{1}{2}\rangle$ use lowering operator again

$$J_-|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \hbar\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\Rightarrow |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \tfrac{1}{\hbar}(S_{1-} + S_{2-})|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle = \tfrac{1}{\hbar}(S_{1-} + S_{2-})\left[\tfrac{1}{\sqrt{3}}\,|0\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1\ -\tfrac{1}{2}\rangle\right]$$

$$= \tfrac{1}{\hbar}\left\{\tfrac{1}{\sqrt{3}}\left(\hbar\sqrt{1(1+1)-0}\ |-1\ \tfrac{1}{2}\rangle + \hbar\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2})}\ |0\ -\tfrac{1}{2}\rangle\right)\right.$$

$$\left. - \sqrt{\tfrac{2}{3}}\left(\hbar\sqrt{1(1+1) - 1(1-1)}\ |0\ -\tfrac{1}{2}\rangle + 0\right)\right.$$

$$= \sqrt{\tfrac{2}{3}}\ |-1\ \tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}\ |0\ -\tfrac{1}{2}\rangle$$

The basis of eigenvectors is

$|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |1\ \tfrac{1}{2}\rangle$

$|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0\ \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1\ -\tfrac{1}{2}\rangle$

$|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0\ -\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|-1\ \tfrac{1}{2}\rangle$

$|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = |-1\ -\tfrac{1}{2}\rangle$

$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}\,|0\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1\ -\tfrac{1}{2}\rangle$

$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|-1\ \tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}\,|0\ -\tfrac{1}{2}\rangle$

d) $H = H_0 + W \qquad W = a\,\vec{S}_1\cdot\vec{S}_2 \qquad \vec{S} = \vec{S}_1 + \vec{S}_2$

$$\vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\,\vec{S}_1\cdot\vec{S}_2$$

$$\Downarrow$$

$$W = \frac{a}{2}\left[\vec{S}^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2}\right]$$

$$\vec{S}_1^{\,2} = S_1(S_1+1)\hbar^2 = 2\hbar^2$$
$$\vec{S}_2^{\,2} = S_2(S_2+1)\hbar^2 = \tfrac{3}{4}\hbar^2$$
$$\vec{S}^2 = S(S+1)\hbar^2$$

$$\Downarrow$$

$$W = \frac{a\hbar^2}{2}\left[S(S+1) - \tfrac{11}{4}\right]$$

$$H|J,M\rangle = (H_0 + W)|J,M\rangle = \underbrace{E_0}_{\text{let } = 0} + \underbrace{W|J,M\rangle}_{\text{solve for these}}$$

$\vec{J} = \vec{S}_1 + \vec{S}_2 = \vec{S}$

$J = 3/2$

$$W|3/2, M\rangle = \frac{a\hbar^2}{2}\left[\tfrac{3}{2}\left(\tfrac{3}{2}+1\right) - \tfrac{11}{4}\right]|3/2, M\rangle$$

$$= \frac{a\hbar^2}{2}\left[\tfrac{15}{4} - \tfrac{11}{4}\right]|3/2, M\rangle$$

$$= \frac{a\hbar^2}{2}|3/2, M\rangle \qquad M = 3/2, 1/2, -1/2, -3/2$$

$$\boxed{E_{3/2} = \frac{a\hbar^2}{2}, \quad 4 \text{ degeneracies}}$$

$J = 1/2$

$$W|1/2, M\rangle = \frac{a\hbar^2}{2}\left[\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{11}{4}\right]|1/2, M\rangle$$

$$= \frac{a\hbar^2}{2}\left[\tfrac{3}{4} - \tfrac{11}{4}\right]|1/2, M\rangle$$

$$= \frac{a\hbar^2}{2}(-2)|1/2, M\rangle$$

$$= -a\hbar^2 |1/2, M\rangle \qquad M = 1/2, -1/2$$

$$\boxed{E_{1/2} = -a\hbar^2 \qquad 2 \text{ degeneracies}}$$

e) Using the expressions for $|\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle$ and $|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$ in terms of the direct pr. basis we can rewrite the initial state in terms of stationary states as:

$$|\psi(0)\rangle = |-1\ \tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

Then the time dependence is:

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{3}} e^{-ia\hbar t/2}|\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{2}{3}}\, e^{ia\hbar t}|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

OK

There is no way this state will evolve under this Hamiltonian into $|1\ {-\tfrac{1}{2}}\rangle$ because it corresponds to different z component, $M = m_1 + m_2$ while J_z is a constant of the motion.